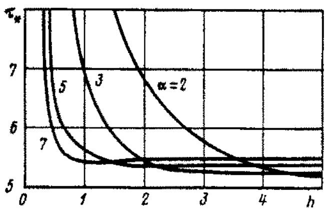
thickness h and the parameter α.

Direct substitution shows that the boundary value problem for the ordinary differential equations is selfconjugate. This is ensured by the existence of a potential for the stresses and the potential character of the external loads. In the case in question the latter condition holds irrespective of the follower character of the load. It can be shown that a uniform tangential load of the type (1.1) distributed over a closed contour is conservative, and this justifies the use of the static Euler method.



The figure shows the results of computations for the neo-Hookean material. The curves corresponding to various values of α characterize the dependence of τ_* on h. We see that on reducing the ring thickness the value of the critical load increases for fixed α. Thus the ring becomes more stable when h decreases. When the load is increased in steps, a form of equilibrium characterized by large values of α occurs within the range of small thicknesses. For thick rings the converse is true. We note that the stability curves have a minimum. On passing the minimum point the loads increase slightly and have a horizontal asymptote as $h \to \infty$.

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EFFECT OF A SMALL DEVIATION IN THE FORM OF THE SHELLS OF REVOLUTION FROM AXIAL SYMMETRY ON THEIR STATE OF STRESS*

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The effect of small, non-axially symmetric imperfections of the middle surface in shells of revolution on their stresses and displacements is studied. A strong dependence on them is found both in statics as well as thermoelasticity. The general theoretical results are confirmed by a numerical study of the displacement and stress fields in cylindrical and conical shells with small imperfections of the type $f(z)\cos m\varphi$.

1. We know that shells with free boundaries (we shall call them free shells) are in, general, compliant, and only weakly resist the action of external loads. However, if the external loads satisfy prescribed integral conditions formulated in the theorem on flexure /1, 2/, the shells become stiff. The stiffness is, however, unstable and vanishes when there are minute deviations from the conditions, whereupon the shell bends and large displacements result. Using the static-geometrical analogy, we find that the problem of analyzing a free shell under external load is equivalent to the problem of computing a shell clamped along its boundary (we shall call it the clamped shell) in a temperature field /3/. From this, we find that, according to the above analogy, the appearance of instability of the stress state in free shells when there are small changes in external load, implies a certain instability in the stress state in clamped shells for small variations in the temperature load.

We will derive asymptotic estimates which will be needed later, for the stress state in free and clamped shells under the action of slowly varying loads of single intensity /4/. In a free shell, when the conditions of the theorem on flexures do not hold, the tangential stress σ_1 and the flexural stress σ_2 are of the order of

$$2Eh\,(u_1,\ u_2,\ w) = h_*^{-2}a^2O\,(\mathbf{R});\ \sigma_1 = h_*^{-1}O\,(\mathbf{R});\ \sigma_2 = h_*^{-2}O\,(\mathbf{R})$$

Here u_1, u_2, w are the displacement vector components, h is the half-thickness of the shell, $\mathbf{R}$ is the external load vector, a is the characteristic linear dimension, E is Young's modulus and $h_* = h/a$ is a small parameter. In a clamped shell we have

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$$2Eh\,(u_1,\,u_2,\,w) = a^2O\,(\mathbf{R}),\ \sigma_1 = h_*^{-1}O\,(\mathbf{R}),\ \sigma_2 = O\,(\mathbf{R})$$

The above estimates show that the stresses in a free shell are flexural, since they are h_*^{-1} -times as large as the tangential stresses. For a given load intensity the displacement in the free shell is h_*^{-2} times as large as that in the clamped shell. If the loads acting on the free shell satisfy the conditions of the theorem on flexures, then the estimates for the displacements and stresses in it are close to those for the clamped shell. The substantial dependence of the stress state in a free shell on the external loads implied by the estimates, can be illustrated as follows. A change in the load intensity by a small amount of the order of h_*, causes a change in the displacement by a quantity of the order of h_*^{-1}, in flexural stresses by a quality of the order of h_*^{-1} times, and in tangential stresses by the quantity h_*°. In a clamped shell the same change in the load will cause a change in the displacements by h_*, in tangential stresses by h_*°, and in flexural stresses by h_*, i.e. the stress state will change by an insignificant amount.

Analogous asymptotic estimates for a free and clamped shells subjected to temperature loads of the same intensity will be, in the case of a free shell,

$$(u_1,\,u_2,\,w) = aO\,(\mathbf{Q}),\ \sigma_1 = Eh_*^2O\,(\mathbf{Q}),\ \sigma_2 = Eh_*O\,(\mathbf{Q})$$

where Q is the external temperature load vector in the equations of continuity of deformations. In the clamped shell we have

$$(u_1,\,u_2,\,w) = aO\,(\mathbf{Q}),\ \sigma_1 = Eh_*^2O\,(\mathbf{Q}),\ \sigma_2 = Eh_*O\,(\mathbf{Q})$$

The methods of estimating the thermoelastic stress state are the same as those used for force loads. The estimates themselves follow from the analogy between the thermoelastic and static problem. At first sight the estimates in the thermoelastic problems appear to be less convincing, with the power indices of the shell parameter found to be positive. We must remember, however, that it is not the absolute, but the relative values of the quantities that are important. We see from the estimates that for a given temperature load intensity the stress state in the clamped shell is h_*^{-1} times as large as that in the free shell. As before we shall have the substantial dependence of the thermoelastic stress state in the clamped shell on the variations in the temperature field. In the free shell this dependence is weak. The estimates obtained for the static case of free shells become invalid, as was said above, when the loads satisfy the conditions of the theorem on flexures.

Let us formulate this theorem for a shell with free edges /4/: under the specified supplementary conditions, the stress state in a free shell will be membrane state of stress if and only if the corresponding external load does no work on the displacements of all possible flexures of the middle surface. For the thermoelastic problems in clamped shells we have a theorem, called in /5/ the theorem on possible static states. The theorem is so called by analogy with the concepts of flexures in free shells and static states in the clamped shells. Let us formulate this theorem: if the temperature loads vector **Q** in a clamped shell is orthogonal to the static states vector (compatible with the tangential boundary conditions if such conditions exist), then a flexural stress state occurs in a shell acted upon by the given temperature field. By the temperature load vector **Q** we mean the vector whose components are represented by the right-hand sides of the equations of continuity of deformations. The temperature load vector in the thermoelastic problem corresponds to the load vector in the static problem. By the static stress vector we mean the vector $\mathbf{V} = (a_1, a_2, c)$, whose components are functions of the stresses a_1, a_2, c, corresponding, according to the static-geometrical analogy, to the displacements u_1, u_2, w.

Thus we see that the free shell only weakly resists the force loads, but resists the temperature loads quite well. The clamped shell on the other hand, resists the action of force quite well but weakly resists that of the temperature. Mathematical by, static stresses in the shell are caused by the lack of homogeneity in the equations of equilibrium, and temperature stresses result from the lack of homogeneity in the equations of continuity. This means that the shell reacts, in principle, differently to the right-hand sides of the equations of equilibrium and of continuity of deformations. We can make the following assertions. The stress state in the clamped (free) shells depends correctly (incorrectly) on the right-hand sides of the equations of equilibrium, and incorrectly (correctly) on the right-hand sides of the equations of continuity of deformations. The correctness or incorrectness in the given case means that small variations in the right-hand sides result in substantial or insubstantial changes in the stress state.

2. Let us now consider the question of how the stress state of a shell of revolution will change if its form departs somewhat from the perfect one. We will assume, although this is not essential, that the shell boundary is perfect. We will describe the imperfections in the shell using functions of the type $\varepsilon f\,(z) \cos m\varphi$ where z is the meridianal and φ the peripheral coordinate, $f\,(z)$ is a slowly varying function and ε is a small quantity. We will write the coefficients and required solution of the system of equations in the form of sums in powers of the small parameter ε. After substituting the coefficients and solution into the initial system, the latter separates into several subsystems of equations. Construction of the zero

approximation represents the solution of the problem for a perfect shell of revolution. Every subsequent approximation is constructed by integrating the system of equations for the perfect shell with different right-hand sides in the equations of equilibrium, as well as in the equations of continuity of deformations. The procedure for constructing the solution described here is realized when computing cylindrical and conical shells with free, as well as with rigidly clamped boundaries. Deviation from axial symmetry is described in terms of the variable distance r between the axis of revolution and the middle surface of the shell, which for a cylindrical shell has the form

$$r(z, \varphi) = R_0 [1 + \varepsilon f(z) \cos m\varphi] \tag{2.1}$$

where R_0 is the radius of the perfect cylinder and m is the number of waves in the peripheral direction. In this case the geometrical characteristics, i.e. their zero and first order terms of expansion into the asymptotic sums, have the form

$$\begin{gathered} A_1 = 1,\ A_2 = r,\ \chi = \pi/2,\ L_{11} = R_0 f'' \varepsilon \cos m\varphi \\ L_{22} = -R_0 [1 + \varepsilon f (m^2 + 1) \cos m\varphi];\ L_{12} = -mR_0 f' \varepsilon \sin m\varphi \end{gathered} \tag{2.2}$$

Here A_1, A_2 and L_{11}, L_{12}, L_{22} are the coefficients of the first and second quadratic forms of the surface, and χ is the angle between the coordinate lines.

Expressions (2.1) and (2.2) were substituted into the equations of equilibrium and continuity of deformations written in any orthogonal coordinate system /4/. All unknown quantities in these equations were written in the form of sums in powers of ε. Separation of the variables is possible, provided that the first approximation is axisymmetric. As a result, the system of equations of equilibrium takes the following form to a first approximation:

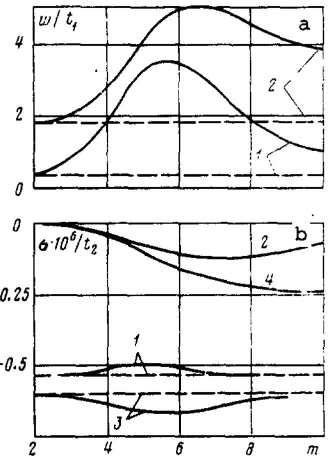
$$\begin{aligned}
&\frac{dT_1^1}{dz} + \frac{m}{R_0} S^1 = r' (T_2^\circ - T_1^\circ) - r \frac{dT_1^\circ}{dz} - R_0 r'' N_1^\circ - r X_1 \\
&\frac{dS^1}{dz} - \frac{m}{R_0} T_2^1 + \frac{1}{R_0} \frac{dH^1}{dz} + \frac{m}{R_0^2} G_2^1 = m r' N_1^\circ \\
&\frac{d^2 G_1^1}{dz^2} + \frac{m^2}{R_0^2} G_2^1 - \frac{2m}{R_0} \frac{dH^1}{dz} + \frac{T_2^1}{R_0} = R_0 r'' T_1^\circ - \frac{r}{R_0} (m^2 - 1) T_2^\circ - \\
&\quad r'' (G_1^\circ - G_2^\circ) - r' \frac{d}{dz} (2G_1^\circ - G_2^\circ) - r \frac{d^2 G_1^\circ}{dz^2} + r \frac{dN_1^\circ}{dz}
\end{aligned}$$

The system of equations of continuity of deformations, to a first approximation, takes the form

$$\begin{aligned}
&\frac{d\gamma_2^1}{dz} - \frac{m}{R_0} \tau^1 = r' (\varkappa_1^\circ - \varkappa_2^\circ) - r \frac{d}{dz} (\varkappa_2^\circ + \varkappa_T) - \\
&\quad R_0 r'' \frac{d}{dz} (\varepsilon_2^\circ + \varepsilon_T) - \frac{m^2 r'}{R_0} (\varepsilon_2^\circ + \varepsilon_T) \\
&\frac{d\tau^1}{dz} + \frac{m}{R_0} \varkappa_1^1 - \frac{1}{R_0} \frac{d\omega^1}{dz} - \frac{m}{R_0^2} \varepsilon_1^1 = m \left[r' \frac{d}{dz} (\varepsilon_1^\circ - \varepsilon_2^\circ) + r'' (\varepsilon_1^\circ + \varepsilon_T) \right] \\
&\frac{\varkappa_1^1}{R_0} + \frac{d^2 \varepsilon_2^1}{dz^2} - \frac{m}{R_0} \frac{d\omega^1}{dz} - \frac{m^2}{R_0^2} \varepsilon_1^1 = R_0 r'' \varkappa_2^\circ - \frac{r (m^2 - 1)}{R_0} \varkappa_1^\circ + \\
&\quad r' \frac{d}{dz} (\varepsilon_1^\circ - 2\varepsilon_2^\circ) + r'' (\varepsilon_1^\circ - \varepsilon_2^\circ) + \left[R_0 r'' - \frac{r (m^2 - 1)}{R_0} \right] \varkappa_T - \\
&\quad r' \frac{d\varepsilon_T}{dz}, \quad \varepsilon_T = \frac{1}{2h} \int_{-h}^{h} \beta T^\circ dz, \quad \varkappa_T = \frac{3}{2h^3} \int_{-h}^{h} \beta T^\circ z\, dz
\end{aligned}$$

Here β, T° is the linear expansion coefficient and the temperature to which the shell material is heated, X_1 is the meridianal load and $\varepsilon_1, \varepsilon_2, \omega, \varkappa_1, \varkappa_2, \tau, T_1, T_2, S, G_1, G_2, H$ are the deformations and their force factors in the notation used in /4/. The superscript indicates the zeroth or first approximation. The elasticity relations remain homogeneous and are therefore not given here. The system was integrated numerically on a computer using a method due to S.K. Godunov /6/.

The static and thermoelastic problems were studied for cylindrical and conical shells with small imperfections. The following dimensionless parameters were chosen for the cylindrical shell: the half-thickness $h_* = h/R_0 = 0.0025$, the relative strength $L/R_0 = 4.0$ (L is the length of the axis of rotation of the shell). For a conical shell the above parameters had the following values: $h_* = h/R_0 = 0.00667$, $L/R_0 = 4.81$ (R_0 is the radius of the smaller base of the cone), and the cone half-angle was $\alpha = 30°$. In each case it was assumed that $\varepsilon = h_*/2$. The computations were carried out separately for the values of m ranging from 2 to 10, in order to study the influence of various types of defects on the stress state. Fig.1 shows the results of the computations for the cylindrical and conical shell. One edge of the shell was clamped, and the other was coupled to a perfectly rigid ring through which the axial force was transmitted.



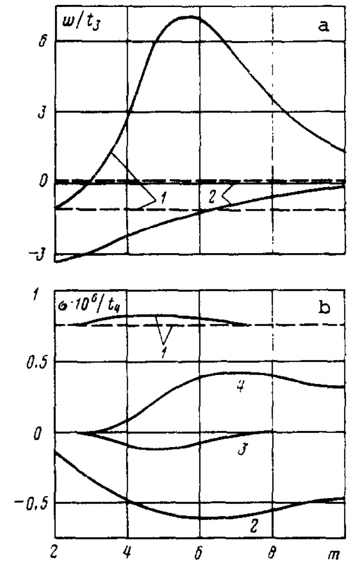


Fig.1 Fig.2

Fig.1a shows the variation in the maximum value of total flexture w with m. All graphs show dimensionless quantities, and the values of the cofficients converting the parameter to dimensionless form are $t_1 = P/(4\pi Eh)$, $t_2 = P/(4\pi R_0 h)$ where P is the axial compressive force. For a cylindrical shell $P = 98.1$ N and for the conical shell $P = 13.1$ N; *1* denotes the curve for the cylindrical shell and *2* for the conical shell (shown with opposite sign). The dashed straight lines show the value of the axisymmetric flexure (zero approximation).

Fig.1b shows the dependence on m of the maximum value of the total stress due to the tangential forces σ_1 (the dashed lines correspond to σ_1 in the zeroth approximation): *1* for the cylindrical and *2* for the conical shell. The same graph shows the dependence on m of the maximum value of the total stress due to the moments σ_2 , *3* for the cylindrical shell and *4* for the conical shell (shown with opposite sign).

Fig.2 shows the results of computations for the cylindrical shell. In the first case the cylindrical shell, rigidly clamped at one edge and coupled to a perfectly rigid ring, was uniformly heated, and in the second case a cylindrical shell with free edges was subjected to a uniform external pressure. Fig.2a shows the variation of w with m. The values of the coefficients converting the parameters to dimensionless form are: $t_3 = \beta T^\circ R_0 = 140 p R_0/E$, $t_4 = E\beta T^\circ = 140p$ where p is the external pressure ($p = 15\cdot 10^6$ N/m^2, $T^\circ = 100^\circ$ C) *1* denotes the curve for a heated cylindrical shell, and *2* is for a cylindrical shell under external pressure. The dashed lines show the value of the axisymmetric flexure.

Fig.2b shows the variation in σ_1 and σ_2 with m. The dashed lines correspond to the value of the parameter in the zeroth approximation; *1, 2* denote curves of σ_1 and σ_2 for a cylindrical heated shell, and *3, 4* curves of σ_1 and σ_2 for a cylindrical shell under external load. The computations show that in all problems the changes in the defects with respect to m substantially affect the displacements of the shell. The flexure, taking the first approximation in ε into account, exceeds, for the defects at $m = 2$, the axisymmetric flexure by a factor of 1.2 for the loaded cylindrical shell, and by a factor of 1.1 for the heated cylindrical shell. The greatest deviation of the flexure from the axisymmetric value was observed for loaded shells, at $m = 5$ and $m = 6$ for the cylindrical and conical shells. For the cylindrical shell the deviation in flexure was found to be 8.5 times that of the axisymmetric flexure, and for the conical shell the value was 2.7 times. For a heated cylindrical shell the maximum value of the flexure was at $m = 6$ equal to 6.6 times that of the axisymmetric flexure. For a free cylindrical shell under external pressure the maximum value at $m = 2$ was 22 times that of the axisymmetric flexure, and the value of w gradually decreased as m increased.

In all cases the change in the defects influences the stress state. The stresses σ_1 due to tangential forces change insignificantly (not more than 10% from the axisymmetric value), while the stresses σ_2 due to the moments increase considerably. For a cylindrical shell under an axial load the increase is 23% of the axisymmetric stress, for a conical shell it is 43%, for a free cylindrical shell under external pressure the increase is 79% and for a heated cylindircal shell σ_2 exceeds σ_1 by a factor of 3.7.

The results confirm the earlier conclusions concerning the incorrectness of the stress state to non-axisymmetric imperfections: the deflection of the form of the middle surface by half the thickness leads to a severalfold change in the displacements. The result is important when studying the strength of shells and in experimental investigations. When measuring the displacements, we must always remember how sensitive they are to the shell imperfections discussed here. Moreover, the imperfections may lead to the appearance in the clamped shell of a flexural stress state comparable with the membrane stress state, i.e. they may change completely in character. The membrane component of the stress state will, in this case, be mainly axisymmetric, and the flexural component will change in accordance with the law governing the change in the imperfections. This may lead, in particular, to the loss of the load carrying capacity of the shell, or to its undesirable deformations. Thus the defects lead to a substantial change in the character of the stress state; in statics and thermoelasticity it changes from the membrane to the mixed stress state.

It should be noted that not all shell imperfections may change the stress state substantially. We can use the theorems on flexure and static states to classify the imperfections and single out the dangerous ones.

We know that in a free shell of revolution the flexures are described in terms of the functions $f_m \cos m\varphi,\ m \geqslant 2$, i.e. there are no flexures axisymmetric and varying as $\cos\varphi$. This means that the axisymmetric loads and loads varying as $\cos\varphi$ both satisfy the conditions of the theorem on flexures, since the work done by these forces on the possible flexures is always equal to zero. Then according to what was said above, when the loads are small, the stresses will always be fairly small. According to the static-geometrical analogy we find exactly the same situation in the case of clamped shells, where again there are no static states axisymmetric and varying as $\cos\varphi$. Therefore the thermoelastic stress will always be fairly small for small temperature difference, axisymmetric and varying as $\cos\varphi$. The picture becomes different if the loads and temperatures in the problems shown vary as $\cos m\varphi,\ m \geqslant 2$. The stresses then increase appreciably even though the intensity of the loads and temperatures is maintained the same. We can generalize this by saying that the imperfections, axisymmetric and varying as $\cos\varphi$, cause insignificant changes in the stress state in clamped as well as free shells of revolution. We can explain this in a different way.

If, in constructing the first approximation we seek the solution using the suggested small-parameter method, the right-hand sides will contain functions axisymmetric or varying as $\cos\varphi$. The first approximation problem will in this case be equivalent to that of computing a shell of revolution with laods that are axisymmetric or vary as $\cos\varphi$, and small temperature loads caused by small imperfections. Since there are neither flexural nor static states present both theorems hold and the dependence on the right-hand sides will be correct: small right-hand sides will give small solutions, i.e. the defects do not cause appreciable changes in the displacements and stresses.

The situation becomes different when the defects vary as $\cos m\varphi,\ m \geqslant 2$. However, in this case we also find that if the right-hand sides of the first approximation formed by the functions of the imperfections are such that the integral conditions of the theorems hold, i.e. the corresponding vector formed by the right-hand sides of the equations of equilibrium is orthogonal to the flexural displacement vector for a free shell, and the vector of the right-hand sides of the continuity of deformations equations is orthogonal to the vector of static states for a clamped shell. Such imperfections are therefore not dangerous, and cause insignificant perturbations in the stress state. The orthogonality requirements may serve as a criterion for estimating the degree of danger introduced by the imperfections.

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